CLOSED-FORM SOLUTION OF A THREE-PHASE VOLTAGE-SOURCE INVERTER FEEDING A THREE-PHASE INDUCTION MOTOR

Ibrahim A. M. ABDEL-HALIM, FIET **Mahmoud A. AL-AHMAR**
iamabd@link.net ma_alahmar2@hotmail.com
Mohammed E. ELFARASKOURY
mohammed_eissa70@yahoo.com
Faculty of Engineering (Shoubra), Benha University, Electrical Engineering Dept. 108 Shoubra St., Cairo, Egypt.

Abstract: *In this paper a closed-form solution of the phase current of a three-phase, voltage-source inverter (VSI) fed induction motor (IM) is presented. Instead of using numerical solution methods, explicit analytical expressions of the steady-state current of the motor are derived.*
To validate the approach presented in this paper the results obtained are compared with previously published numerical results dealing with static R-L loads; considering the induction motor as equivalent to a R-L load, and the two sets of results are found to be almost identical.
Keywords: *Closed-form solution, Voltage-source inverter, Induction motor, Steady-state analysis, numerical solution.*

1. Introduction

The voltage-source inverters are commonly used in speed control of three-phase induction motors from a DC supply. By controlling the switching elements in the voltage-source inverter the applied voltage to the induction motor can be controlled.

A lot of attention was given to voltage-source inverters, control of three-phase induction motors and voltage-source inverters /induction motor systems [1-21].

In previous investigations, numerical methods were used to obtain the transient performance of the system [22-27], and the steady-state performance was obtained by passing through the transient performance first, and were not obtained directly.

In this paper, direct analytical closed-form expressions, at steady-state and at fundamental frequency, for the currents of the VSI/IM system are derived.

To check the validity of the presented approach the results obtained are compared with previously published results for static R-L loads by considering the induction motor , at any speed , as corresponding to a static R-L load .

2. Method of analysis

At steady state, and any speed, the equivalent circuit per phase of the induction motor is as shown in Fig. 1. This equivalent circuit of the induction motor can be presented by an equivalent series R-L load. If the corresponding equivalent resistance is R_e , and the corresponding equivalent reactance is X_e , then:

$$R_e = R_1 + \frac{X_m^2 R_2^{\backslash} s}{R_2^{\backslash 2} + s^2 (X_m + X_2^{\backslash})^2} \quad (1)$$

and

$$X_e = X_1 + \frac{R_2^{\backslash 2} + s^2 X_2^{\backslash} (X_m + X_2^{\backslash})}{R_2^{\backslash 2} + s^2 (X_m + X_2^{\backslash})^2} . X_m \quad (2)$$

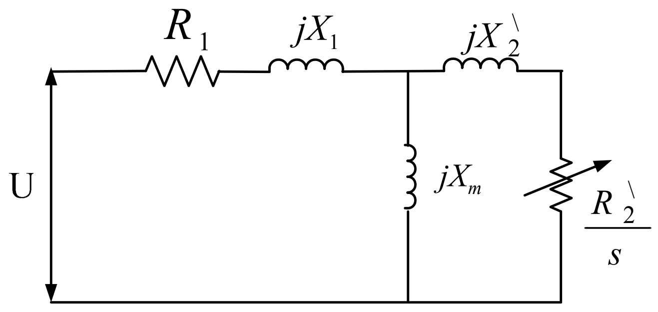

Fig. 1. Induction motor equivalent circuit.

If the phase angle of the motor at any slip, s, is assumed to be φ, then:

$tan\ (\varphi) = X_e / R_e$ (3)

A voltage-source inverter feeding a three-phase induction motor is shown in Fig. 2. In this circuit the switching elements (S) may be either thyristors or transistors. Each switching element will conduct for 180 ° during a cycle.

The differential equation of the line current of the motor for line (a) is:

$$u_a(\omega t) = R_e i_a + L_e \frac{di_a}{dt} \quad (4)$$

where R_e is the input equivalent resistance of the equivalent circuit of the induction motor and L_e is the inductance corresponding to X_e at fundamental frequency.

The inverter six-step voltage waveform, u_a, applied to phase (a) of the motor is shown in Fig. 3, together with a typical waveform of the phase current [1, 7, 22, 28].

Thus, solution of eqn. (4) in the period $0 \leq \omega t \leq \pi/3$, in which $u_a\ (\omega\ t) = U_d / 3$, will be:

$$i_{a1}(\omega t) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}]e^{-\omega t \cot\phi}$$

$$0 \leq \omega t \leq \pi/3 \quad (5)$$

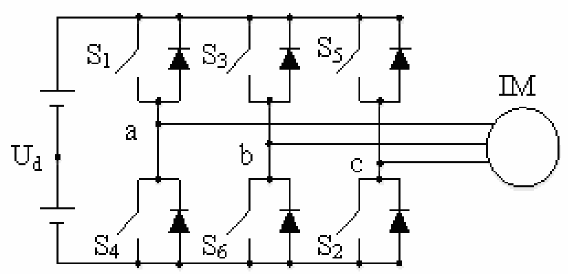

Fig. 2. Voltage-source inverter/induction motor system.

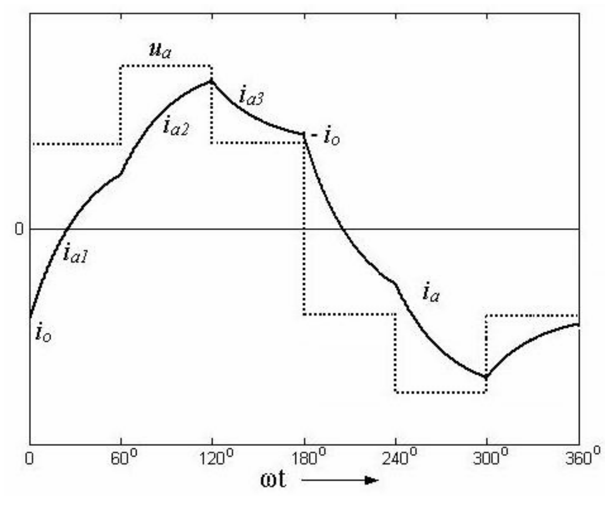

Fig. 3. Voltage-source inverter output phase (a) voltage and current.

where:
i_o is the initial value of $i_a(\omega t)$ at $\omega t = 0$, and its derivation will be given in Section (2.1), and its expression is as given in eqn. (14).
U_d is the inverter input DC voltage.
φ phase angle of the induction motor at fundamental frequency, i.e. $\varphi = tan^{-1}(\omega L_e/R_e)$.
Eqn. (5) can be rewritten as:

$$i_{a1}(\omega t) = k_1 + [\frac{k_1(k_2 - 2)}{k_2^2 - k_2 + 1}]e^{-\omega t\cot\phi} \qquad 0 \le \omega t \le \pi/3 \qquad (6)$$

where: $k_1 = U_d/(3R_e)$ and $k_2 = e^{-(\pi/3)\cot\varphi}$

For the period $\pi/3 \le \omega t \le 2\pi/3$, in which $u_a(\omega t) = 2U_d/3$, Fig. 3, solution of eqn. (4) is obtained as:

$$i_{a2}(\omega t) = \frac{2U_d}{3R_e} + [i_{a1}(\pi/3) - \frac{2U_d}{3R_e}]e^{-(\omega t - (\pi/3))\cot\phi} \qquad \pi/3 \le \omega t \le 2\pi/3 \qquad (7)$$

where $i_{a1}(\pi/3)$ is obtained from eqn. (5) with $\omega t = \pi/3$. Thus:

$$i_{a1}(\pi/3) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}]e^{-(\pi/3)\cot\phi}$$

Therefore, eqn. (7) becomes:

$$i_{a2}(\omega t) = \frac{2U_d}{3R_e} + [i_o - \frac{U_d}{3R_e}e^{(\pi/3)\cot\phi} - \frac{U_d}{3R_e}]e^{-\omega t\cot\phi} \qquad \pi/3 \le \omega t \le 2\pi/3 \qquad (8)$$

Eqn. (8) can be rewritten as:

$$i_{a2}(\omega t) = 2k_1 - [\frac{k_1(k_2 + 1)}{k_2(k_2^2 - k_2 + 1)}]e^{-\omega t\cot\phi} \qquad \pi/3 \le \omega t \le 2\pi/3 \qquad (9)$$

For the period $2\pi/3 \le \omega t \le \pi$, in which $u_a(\omega t) = U_d/3$, Fig. 3, solution of eqn. (4) becomes:

$$i_{a3}(\omega t) = \frac{U_d}{3R_e} + [i_{a2}(2\pi/3) - \frac{U_d}{3R_e}]e^{-(\omega t - (2\pi/3))\cot\phi} \qquad 2\pi/3 \le \omega t \le \pi \qquad (10)$$

where $i_{a2}(2\pi/3)$ is obtained from eqn.(8) at $\omega t = 2\pi/3$. Thus:

$$i_{a2}(2\pi/3) = i_o e^{-(2\pi/3)\cot\varphi} + \frac{U_d}{3R_e}[2 - e^{-(\pi/3)\cot\varphi} - e^{-(2\pi/3)\cot\varphi}]$$

$$i_{a2}(2\pi/3) = i_o e^{-(2\pi/3)\cot\phi} + \frac{U_d}{3R_e}[2 - e^{-(\pi/3)\cot\phi} - e^{-(2\pi/3)\cot\phi}]$$

Therefore, eqn. (10) becomes:

$$i_{a3}(\omega t) = \frac{U_d}{3R_e} + [i_o e^{-(2\pi/3)\cot\phi} + \frac{U_d}{3R_e}(2 - e^{-(\pi/3)\cot\phi} - e^{-(2\pi/3)\cot\phi}) - \frac{U_d}{3R_e}]e^{-(\omega t - 2\pi/3)\cot\phi} \qquad 2\pi/3 \le \omega t \le \pi \qquad (11)$$

Eqn. (11) can be rewritten as:

$$i_{a3}(\omega t) = k_1 + [\frac{k_1(1 - 2k_2)}{k_2^2(k_2^2 - k_2 + 1)}]e^{-\omega t\cot\phi} \qquad 2\pi/3 \le \omega t \le \pi \qquad (12)$$

Thus, during the first positive half cycle of the phase voltage applied to the motor, Fig.(3), the expression of the waveform of the steady-state current of phase (a) of the motor will be:

$i_a = i_{a1}$ $\quad 0 \le \omega t \le \pi/3$

$i_a = i_{a2}$ $\quad \pi/3 \le \omega t \le 2\pi/3$

$i_a = i_{a3}$ $\quad 2\pi/3 \le \omega t \le \pi$

i_{a1}, i_{a2} and i_{a3} are obtained from eqns. (6), (9) and (12) respectively.

2.1 Determination of the initial current i_o

Since, the current of $i_{a1}(\omega t)$ at $\omega t = 0$ is i_o and the current of $i_{a3}(\omega t)$ at $\omega t = \pi$ is $(-i_o)$, Fig. 3, thus:

$$i_{a3}(\pi) = \frac{U_d}{3R_e} + [i_o - \frac{U_d}{3R_e} e^{(\pi/3)\cot\phi} + \frac{U_d}{3R_e} e^{(2\pi/3)\cot\phi} - \frac{U_d}{3R_e}] e^{-\pi\cot\phi} = -i_o$$

Thus:

$$i_o = \frac{\frac{U_d}{3R_e}(e^{-\pi\cot\varphi} + e^{-(2\pi/3)\cot\varphi} - e^{-(\pi/3)\cot\varphi} - 1)}{(1 + e^{-\pi\cot\varphi})} \quad (13)$$

Eqn. (13) can be rewritten as:

$$i_o = \frac{k_1(k_2^3 + k_2^2 - k_2 - 1)}{(k_2^3 + 1)} \quad (14)$$

With the initial current i_o determined using eqn. (13) or eqn. (14), for certain phase angle of the motor, φ, direct and explicit form expressions of the steady-state induction motor current can be obtained.

(2.2) Determination of the motor rms Current

The rms current can be obtained, Fig. (3), from:

$$I_a = [\frac{1}{\pi}\{ \int_0^{\pi/3} i_{a1}^2(\omega t)\, d\omega t + \int_{\pi/3}^{2\pi/3} i_{a2}^2(\omega t)\, d\omega t + \int_{2\pi/3}^{\pi} i_{a3}^2(\omega t)\, d\omega t \}]^{0.5} \quad (15)$$

Using the expressions of i_{a1}, i_{a2} and i_{a3} given in eqns. (6), (9) and (12) respectively, the rms current of the motor (I_a) can be obtained from eqn. (15) as:

$$I_a = [\frac{1}{\pi}\{2\pi k_1^2 + 6k_1 i_0 \tan(\phi) - \frac{3i_0^2 \tan(\phi)(k_2^2 - k_2 + 1)}{(k_2^2 - 1)}\}]^{0.5} \quad (16)$$

(2.3) Input equivalent resistance and reactance of the motor

For a certain value of the phase angle of the induction motor, φ, and using eqns.(1) and (2):

$\tan(\varphi) = X_e / R_e$

Thus :

$X_e = R_e \tan(\varphi)$

and the following expression relating the parameters of the motor with slip, s, and φ can be obtained as:

$$A s^2 + B s + C = 0 \quad (17)$$

where:

$A = (X_m + X_2^{\backslash})[(X_1 X_m + X_1 X_2^{\backslash} + X_2^{\backslash} X_m - (R_1 X_m + R_1 X_2^{\backslash}) \tan(\varphi)]$

$B = X_m^2 R_2^{\backslash} \tan(\varphi)$

$C = (R_2^{\backslash})^2 [X_1 + X_m - R_1 \tan(\varphi)]$

The slip, s, can be obtained from the solution of eqn. (17) as:

$$s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

(2.4) Relationship between motor and static R-L load currents

The impedance of the motor can be expressed by :

$$Z_m = R_e \sqrt{1 + (\tan\varphi_m)^2} \quad (18)$$

and that of the static R-L load can be expressed by:

$$Z = R\sqrt{1 + (\tan\phi_{RL})^2} \quad (19)$$

Thus, the relationship between the rms current of the motor, I_m, and the rms current of the static R-L load, I_{RL}, for the same applied voltage , will be:

$$\frac{I_m}{I_{RL}} = \frac{R\sqrt{1 + \tan(\phi_{RL})^2}}{R_e\sqrt{1 + \tan(\phi_m)^2}} \quad (20)$$

Thus, if the phase angle of the motor, φ_m , is equal to the phase angle of the static R-L load, φ_{RL}, the relationship between I_m and I_{RL} will be :

$$\frac{I_m}{I_{RL}} = \frac{R}{R_e} \tag{21}$$

i.e. at the same phase angle:

$$I_m = \frac{R}{R_e} I_{RL} \tag{22}$$

3. Results

In order to obtain the results, for the presented approach, for the current of phase (a) of the motor, as an example, MATLAB programming package is used to obtain the results at steady-state and at fundamental frequency when the three-phase voltage-source inverter is feeding the three-phase induction motor whose parameters are given in the Appendix.

To check the validity of the proposed approach, the obtained results are compared with previously published results obtained for static R-L loads [28]. In reference [28] results were obtained numerically and analytically when a VSI is feeding a static R-L load.

Fig. 4 shows the instantaneous values of the phase current i_a for the case of the motor at a slip, *s*, that will give a phase angle $\varphi = 30^o$ and an input impedance of $Z_m = 10\ \Omega$. In this case eqn. (1) is used to obtain the corresponding slip of the motor, which was obtained as s=0.0413.
The instantaneous current of the motor is compared with the current of a static R-L load whose $\varphi_{RL} = 30^o$, and its impedance is $Z = 10\ \Omega$.
It is evident from this figure, Fig. 4, that the results of the motor current and the static R-L load current are almost identical, which proves the validity of the approach presented .

The rms current of the motor at any phase angle, φ_m, can be obtained using the rms current value of the static R-L load, at the same phase angle and applied voltage, employing eqn.(22).

In eqn. (22) the equivalent resistance of the motor R_e at certain slip is required.
To obtain the slip corresponding to a certain phase angle, φ_m, of the motor, Fig. 5, which relates the motor power factor with its slip, is used. Fig. 5 is obtained using eqns. (1), (2) and (3). For the range of the slip considered (0 to 1), for each value of the slip, s, eqns. (1) and (2) are used to obtain the corresponding values of R_e and X_e, and the corresponding phase angle is obtained from eqn. (3). For a phase angle for the static R-L load as that of the motor and for any value of its impedance, Z, and using the same applied voltage as that applied to the motor, its rms current can be obtained. Thus, to obtain the motor rms current at the same phase angle as that of the static R-L load and at the same applied voltage, eqn. (22) can be used directly.

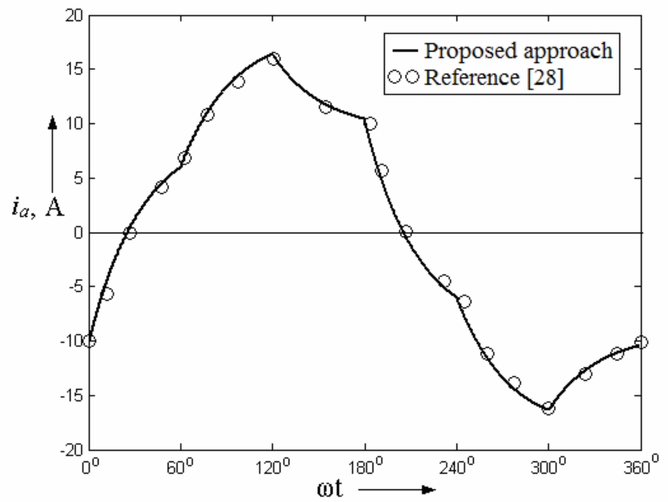

Fig. 4. Phase (a) current waveform.

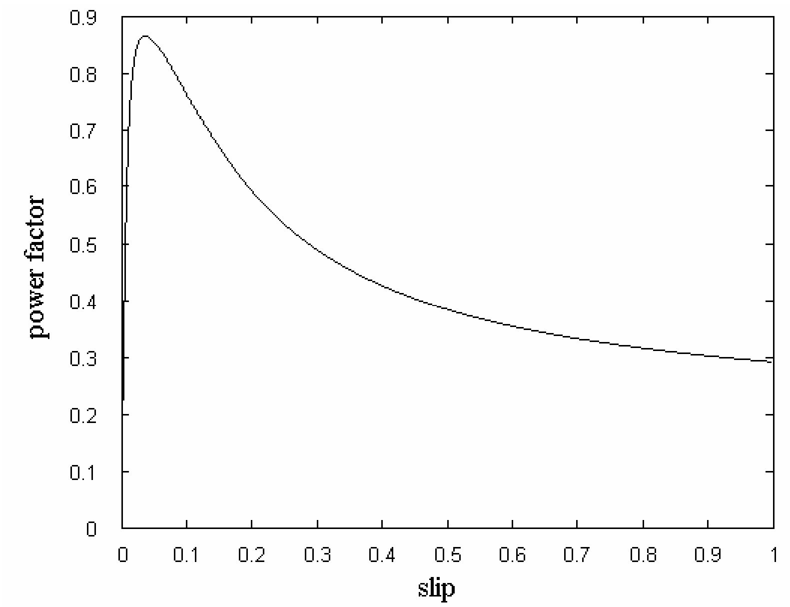

Fig. 5. Motor power factor/ slip characteristic.

Fig. 6 shows a comparison between the rms values of the currents of the motor and of the static R-L load, when fed from three-phase voltage-source inverters, for different values of phase angles. From this figure it is evident the two sets of results are in very close agreement.

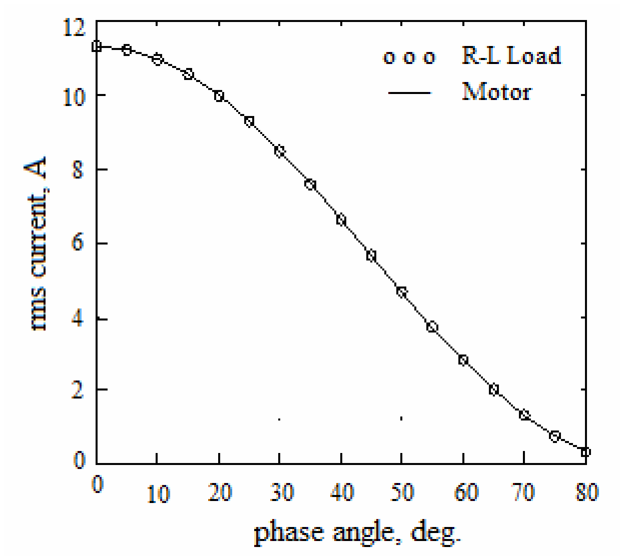

Fig. 6. Comparison between rms values of the motor current and the static R-L current.

4. Conclusions

In this paper, closed-form analytical solutions for currents at steady-state and fundamental frequency for the three-phase induction motor fed from a three-phase voltage-source inverter are presented. These explicit closed-form expressions of the motor current can be used to obtain the performance characteristics of the system.

The results obtained using these analytical closed-form expressions were compared with those obtained analytically and numerically in previously published reference dealing with three-phase static R-L load fed from a three-phase voltage-source inverter, and the two sets of results were in very close agreement.

List of symbols

R_1, $R_2^\backslash$ induction motor stator and rotor resistances per phase respectively referred to stator.
X_1, $X_2^\backslash$ induction motor stator and rotor leakage reactances per phase respectively referred to stator.
X_m induction motor magnetizing reactance per phase.
U induction motor terminal voltage per phase.
s induction motor slip.
ω fundamental angular frequency of the supply.

References

1. De Four, R. Wadi, T.: *Natural Space Vector Modulation in Six-Step Voltage Source Inverters.* In: International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering, Vol. 3, Issue 7, 2014, pp. 10409-10415.
2. Behera, P.K., Behera, M.K., Sahoo, A.K.: *Comparative Analysis of Scalar & Vector Control of Induction Motor through Modeling & Simulation*. In: International Journal of Innovative Research in Electrical, Electronics, Instrumentation and Control Engineering, Vol. 2, Issue 4, 2014, pp.1340-1344.
.3. Singh, J.: *Transient Analysis of Z-Source Inverter Fed Three-Phase Induction Motor Drive by Using PWM Technique*. In: International Electrical Engineering Journal (IEEJ), Vol. 4, No. 1, 2013, pp. 856-863.
4. Manuel, A., Francis, J.: *Simulation of Direct Torque Controlled Induction Motor Drive by Using Space Vector Pulse Width Modulation for Torque Ripple Reduction*. In: International Journal of Advanced Research in Electrical, Electronics and Instrumentation Engineering, Vol. 2, Issue 9, 2013, pp. 4471-4478.
5. Sinha, A. N., Minz, S.: *Analysis of Vector Control of Induction Motor Using Matlab and Its Application in Traction System*. In: VSRD International Journal of Electrical, Electronics and Communication Engineering, Vol. III, Issue X, 2013, pp. 451-456.
6. El-saady, G., El-Nobi, A. Ibrahim, Elbesealy, M.: *V/F Control of Three Phase Induction Motor Drive with Different PWM Techniques*. In: Innovative Systems Design and Engineering, Vol.4, No.14, 2013, pp. 131-144.
7. Abdel-Halim, I. A. M., Hamed, G., Hassan, A. M.: *Steady-State Performance of a Directly Connected PV Array/Six-Step VSI/ Induction Motor System*. In: International Journal of Electrical and Power Engineering, Vol. 4, No. 2, 2010, pp. 105-112.
8. Ogbuka, C.U., Agu, M.U.: *A Modified Closed Loop V/F Controlled Induction Motor Drive*. In: The Pacific Journal of Science and Technology, Vol. 10. No.1, 2009, pp. 52-58.
9. Klima, J., Chomat, M., Schreier, L.: *Analytical Closed-Form Investigation of PWM Inverter Induction Motor Drive Performance under DC Bus Voltage Pulsation*. In: IET Electr. Power Appl., Vol. 2, No. 6, 2008, pp. 341–352.
10. Pao-La-Or, P., Sujitjorn, S., Kulworawanichpong, T., Peaiyoung, S.: *Studies of Mechanical Vibrations and Current Harmonics in Induction Motors Using Finite Element Method.* In: WSEAS Transactions on Systems, Vol. 7, Issue 3, 2008, pp. 195-202.
11. Abdel-Kader, F.M., El-Saadawy, A., Kalas, A. E., El-Baksawi, O.M.: *Study in Direct Torque Control Of Induction Motor by Using Space Vector Modulation*. In: MEPCON 2008, 12th International Middle-East Power Systems Conference, 2008, pp. 224-229.
12. Abdul Wahab, H.F., Sanusi, H.: *Simulink Model of Direct Torque Control of Induction Machine*. In: American Journal of Applied Sciences, Vol. 5, Issue 8, 2008, pp. 1083-1090.
13. Swamy, R. L., Kumar, P. S.: *Speed Control of Space Vector Modulated Inverter Driven Induction Motor*. In: Proceedings of the International Multi Conference of Engineers and Computer Scientists, Vol. 2, 2008.
14. Klima, J.: *Analytical Investigation of an Induction Motor Fed from Four- Switch VSI with a New Space Vector Modulation Strategy*. In: IEEE Transactions on Energy Conversion, Vol. 21, No. 4, 2006, pp. 832-838.

15. Meziane, S., Toufouti, R., Benalla, H.: *Review of Direct Torque and Flux Control Methods for Voltage Source Inverter Fed Induction Motor.* In: ACSE Journal, Vol.6, Issue 3, 2006, pp. 47-53.
16. Klima, J.: *Closed-Form Analytical Investigation of an Induction Motor Drive Fed from Four-Switch Inverter Six-Step Operation Mode*. In: IEEE, Power Electronics and Drives Systems, PEDS, Vol. 1, 2005, pp. 84-89.
17. Buja, G. S., Kazmierkowski, M. P.: *Direct Torque Control of PWM Inverter-Fed AC Motors—A Survey*. In: IEEE Transactions on Industrial Electronics, Vol. 51, No. 4, 2004, pp. 744-757.
18. Klima, J.: *Analytical investigation of an induction motor drive under inverter fault mode operations*. In: IEE Proc.-Electr. Power Appl., Vol. 150, No. 3, 2003, pp. 255-262.
19. Klima, J.: *Mixed p-z Approach for Analytical Analysis of an Induction Motor Fed from Space-Vector PWM Voltage Source Inverter*. In: ETEP, Vol. 12, No. 6, 2002, pp. 403-413.
20. Silva, N. M., Martins, A. P., Carvalho, A. S.: *Torque and Speed Modes Simulation of A DTC-Controlled Induction Motor.* In: Proceedings of the 10th Mediterranean Conference on Control and Automation – MED2002 Lisbon, Portugal, 2002.
21. Mei, CG., Panda, SK., JX, Xu, Lim, KW.: *Direct Torque Control of Induction Motor - Variable Switching Sectors*. In: IEEE International Conference on Power Electronics and Drive Systems, PEDS-99, 1999, Hong Kong.
22. Wang, T., Fang, F., Wu, X., Jiang X.: *Novel Filter for Stator Harmonic Currents Reduction in Six- Step Converter Fed Multiphase Induction Motor Drives*. In: IEEE Transactions on Power Electronics, Vol. 28, No. 1, 2013, pp. 498- 506.
23. Panchal, S. N., Sheth, V. S., Pandya, A. A.: *Simulation Analysis of SVPWM Inverter Fed Induction Motor Drives*. In: International Journal of Emerging Trends in Electrical and Electronics (IJETEE), Vol. 2, Issue 4, 2013, pp. 18-24.
24. Chaturvedi, P., Dubey, A.: *Analysis of Space Vector Pulse Width Modulation VSI Induction Motor On Various Conditions.* In: International Journal of Application or Innovation in Engineering & Management (IJAIEM), Vol. 2, Issue 12, 2013, pp. 409-414.
25. Soni, P., Burse, K.: *Analysis of Voltage Source Inverters using Space Vector PWM for Induction Motor Drive*. In: IOSR Journal of Electrical and Electronics Engineering, Vol. 2, Issue 6, 2012, pp. 14-19.
26. Klima, J.: *Time and Frequency Domain Analysis of Fault-Tolerant Space Vector PWM VSI-Fed Induction Motor Drive*. In: IEE Proc.-Electr. Power Appl., Vol. 152, No. 4, 2005, pp. 765-774.
27. Abdulrahman, S. M., Kettleborough, J. G., Smith, I. R.: *Fast Calculation of Harmonic Torque Pulsations in a VSI/Induction Motor Drive*. In: IEEE Transactions on Industrial Electronics, Vol. 40, No. 6, 1993, pp. 561-568.
28. Abdel-Halim, I. A. M., Hamed, H. G., Salama, M. N.: *Closed-form solution of a three-phase VSI feeding a three-phase static inductive load.* In: ETEP, Vol.5, No.4, 1995, pp. 279- 284.

Appendix

The parameters of the induction motor used in this system are [12]:
Three-phase induction motor, 7500-W, 400-V, 50-Hz, 16-A, 4-pole with the following parameters:
$R_1 = 0.6\ \Omega$, $X_1 = 0.9425\ \Omega$, $R_2^{\backslash} = 0.4\ \Omega$, $X_2^{\backslash} = 2.325\ \Omega$, $X_m = 37.7\ \Omega$.